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# TORSION OF AN AXISYMMETRIC ANISOTROPIC BODY WITH MIXED BOUNDARY CONDITIONS ON THE SIDE SURFACE

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Differential and integral operators are used to solve the nonsymmetric system of equations characterizing the pure torsion of a body of revolution with variable shear moduli. The stress and displacement functions are expressed by convergent series containing two arbitrary analytic functions of a complex variable and the coefficients of a real argument defined in terms of the shear modulus. As an illustration, the problem of torsion of a hollow cylinder with mixed boundary conditions is considered. The torsion of isotropic rods has been examined in detail in [1], and for anisotropic bodies of revolution in [2, 3].

**1. Initial equations.** The pure torsion of a body of revolution whose axis of cylindrical inhomogeneous anisotropy coincides with the geometric body axis is characterized in the cylindrical coordinates $r\ z\ \theta$ by a linear system of partial differential equations of elliptic type [2]

$$\frac{\partial\varphi}{\partial r} - P(r)\frac{\partial\psi}{\partial z} = 0, \qquad \frac{\partial\varphi}{\partial z} + Q(r)\frac{\partial\psi}{\partial r} = 0$$
$$P(r) = r^3 G_1(r), \qquad Q(r) = r^3 G_2(r) \tag{1.1}$$

Here $\varphi$ is the stress function, $\psi$ is the displacement function, $G_{z\theta} = G_1(r)$, $G_{r\theta} = G_2(r)$ are the shear moduli of the corresponding planes which we consider given (or found from experiment), bounded in a range of variation, and piecewise-continuous functions of the single variable $r$. Two stress components $\tau_{z\theta} = \tau_1(r, z)$, $\tau_{r\theta} = \tau_2(r, z)$ and the tangential displacement $u_\theta = v(r, z)$ defined by the formulas

$$\tau_1 = \frac{1}{r^2}\frac{\partial\varphi}{\partial r} = rG_1(r)\frac{\partial\psi}{\partial z}, \quad \tau_2 = -\frac{1}{r^2}\frac{\partial\varphi}{\partial z} = rG_2(r)\frac{\partial\psi}{\partial r}, \quad v = r\psi \tag{1.2}$$

are not zero in torsion of such bodies. In contrast to [1, 2], let us seek the solution of the system (1.1) in the form of the operator

$$\varphi = \operatorname{Im} \sum_{k=0}^{\infty} \alpha_k(r) w_k(\zeta), \qquad \psi = \operatorname{Re} \sum_{k=0}^{\infty} \beta_k(r) w_k(\zeta) \tag{1.3}$$

$$\zeta = \rho + iz, \qquad \rho = \int \sqrt{\frac{G_1(r)}{G_2(r)}}\, dr \tag{1.4}$$

The real coefficients $\alpha_k$, $\beta_k$ depend only on the single variable $r$, and the analytic functions $w_k(\zeta)$ of the complex argument $\zeta = \rho + iz$ will be selected so that the operator (1.3) would satisfy the system (1.1). Let us introduce appropriate derivatives of the functions (1.3) into the system (1.1). We hence arrive at the two equations

$$\operatorname{Im} \sum_{k=0}^{\infty} \left[\alpha_k' w_k + \left(\alpha_k \sqrt{\frac{G_1}{G_2}} + P\beta_k\right) w_k'\right] = 0 \tag{1.5}$$

$$\operatorname{Re} \sum_{k=0}^{\infty} \left[Q\beta_k' w_k + \left(Q\beta_k \sqrt{\frac{G_1}{G_2}} + \alpha_k\right) w_k'\right] = 0$$

The known relationships $\operatorname{Re} iF(\zeta) = -\operatorname{Im} F(\zeta)$ and $\operatorname{Im} iF(\zeta) = \operatorname{Re} F(\zeta)$ are hence taken into account. The system (1.5) can be satisfied by two means.

**2. Solution in the form of a differential operator.** The system (1.5) is satisfied identically for an arbitrary analytic function $w_0 = w(\zeta)$ if the conditions

$$\alpha_0' = \beta_0' = 0, \qquad \alpha_k' + P\beta_{k-1} + \alpha_{k-1}\sqrt{\frac{G_1}{G_2}} = 0$$

$$Q\beta_k' + \alpha_{k-1} + Q\beta_{k-1}\sqrt{\frac{G_1}{G_2}} = 0, \qquad w_k = w_{k-1} \qquad (k = 1, 2, \ldots)$$

are imposed on the coefficients $\alpha_k$, $\beta_k$ and on the function $w_k(\zeta)$. We hence find

$$\alpha_0 = \alpha = \text{const}, \qquad \beta_0 = \beta = \text{const}, \qquad w_k = w^{(k)}$$

$$\alpha_k = \alpha_k^\circ - \int \left(P\beta_{k-1} + \sqrt{\frac{G_1}{G_2}}\,\alpha_{k-1}\right) dr \tag{2.1}$$

$$\beta_k = \beta_k^\circ - \int \left(\frac{\alpha_{k-1}}{Q} + \sqrt{\frac{G_1}{G_2}}\,\beta_{k-1}\right) dr \qquad (k = 1, 2, \ldots)$$

($\alpha_k^\circ$, $\beta_k^\circ$ are arbitrary constants of integration). The solution (1.3) hence takes the form of a differential operator analogous to that presented in [4]

$$\varphi = \varphi_1 = \operatorname{Im} \sum_{k=0}^{\infty} \alpha_k(r) w^{(k)}(\zeta), \quad \psi = \psi_1 = \operatorname{Re} \sum_{k=0}^{\infty} \beta_k(r) w^{(k)}(\zeta) \tag{2.2}$$

The complex conjugate argument $\bar{\zeta} = \rho - iz$, as well as arguments of the form $\zeta_1 = i\zeta$, $\bar{\zeta}_1 = i\bar{\zeta}$ can be considered in place of $\zeta$ in (2.2). In these cases the coefficients (2.1) are expressed by rather different formulas.

**3. Solution in the form of an integral operator.** The system (1.5) can also be satisfied in a different way for an arbitrary analytic function of the complex variable $w_0 = f(\zeta)$. To do this we impose conditions of the form

$$a_0 \sqrt{\frac{G_1}{G_2}} + Pb_0 = 0, \qquad Qb_0 \sqrt{\frac{G_1}{G_2}} + a_0 = 0$$

$$a_k \sqrt{\frac{G_1}{G_2}} + Pb_k + a'_{k-1} = 0, \quad Qb_k \sqrt{\frac{G_1}{G_2}} + a_k + Qb'_{k-1} = 0 \qquad (3.1)$$

$$f_k'(\zeta) = f_{k-1}(\zeta) \qquad (k = 1, 2, \ldots)$$

on the functions $\alpha_k = a_k(r), \beta_k = b_k(r)$ and $w_k = f_k(\zeta)$ (for convenience we have used the same notation). We hence arrive at the equations

$$f_k = \sum_{m=1}^{k} \frac{C_m}{(k-m)!} r^{k-m} + \iint \ldots \int f(\zeta)\, d\zeta\, d\zeta \ldots d\zeta_k \qquad (3.2)$$

$$a_0 + \sqrt{PQ}\, b_0 = 0$$

$$a'_{k-1} - \sqrt{PQ}\, b'_{k-1} = 0 \qquad (k = 1, 2, \ldots) \qquad (3.3)$$

($C_m$ are arbitrary constants of integration). Without limiting the generality, let us set $C_m = 0$ ($m = 1,2,\ldots k$). Then the solution (1.3) takes the form of an integral operator analogous to the Bergman operator [5] [see also [4])

$$\varphi = \varphi_2 = \operatorname{Im} \sum_{k=0}^{\infty} a_k(r) \int f(\zeta)\, d\zeta^k, \quad \psi = \psi_2 = \operatorname{Re} \sum_{k=0}^{\infty} b_k(r) \int f(\zeta)\, d\zeta^k \quad (3.4)$$

Here the provisional notation of a $k$-tuple integral

$$I_k(\zeta) = \int f(\zeta)\, d\zeta^k = \iint \ldots \int f(\zeta)\, d\zeta\, d\zeta \ldots d\zeta_k \qquad (3.5)$$

is used. For $k = 0$ the integral in (3.5) is absent, and $I_0 = f(\zeta)$, where $f(\zeta)$ is an arbitrary analytic function of complex argument. In the general case the functions $f(\zeta)$ and $w(\zeta)$ are independent. The coefficients $a_k$, $b_k$ in (3.4) are found from (3.3). As a result of integration we obtain

$$a_0 = -D_0 (r^6 G_1 G_2)^{1/4}, \qquad b_0 = D_0 (r^6 G_1 G_2)^{-1/4}$$

$$a_k = -Qb'_{k-1} - (PQ)^{1/4} \left[ D_k - \frac{1}{2} \int \frac{(Qb'_{k-1})'}{(PQ)^{1/4}}\, dr \right]$$

$$b_k = (PQ)^{-1/4} \left[ D_k - \frac{1}{2} \int \frac{(Qb'_{k-1})'}{(PQ)^{1/4}}\, dr \right] \qquad (k = 1, 2, \ldots) \qquad (3.6)$$

($D_k$ ($k = 0,1,2,\ldots$) are arbitrary constants of integration). By analogy with [4, 5], it can be proved that if $w(\zeta)$ and $f(\zeta)$ are bounded in some domain, then the series in the solutions (2.2) and (3.4) converge absolutely and uniformly in the same domain. We do not examine this here.

Solutions in the form (2.2) and (3.4) are mutually independent. A linear integro-differential combination of these solutions is also a solution of the system (1.1). The coefficients in (2.1) and (3.6) are expressed analytically or are tabulated depending on the parameters $G_1(r)$ and $G_2(r)$. By analogy with [4] it can be shown that the coefficients $\overset{\circ}{\alpha}_k$, $\overset{\circ}{\beta}_k$, $D_k$ are not essential. Then, without limiting the generality, it is sufficient to limit oneself to particular solutions, i.e. to set $\alpha_n^\circ = \beta_n^\circ = D_n = 0$ ($n = 1, 2, \ldots$), which will also be considered later.

For example, let us examine the case when the shear moduli are expressed by the power-law functions

$$G_1 = g_1 r^p, \qquad G_2 = g_2 r^q \tag{3.7}$$

where $g_1$, $g_2$ are fixed constants, and $p$, $q$ are any given numbers. Furthermore, setting $\alpha = 0$, $\beta = 1$ in (2.1) for simplicity, we find

$$\alpha_1 = -g_1 \frac{r^{p+4}}{p+4}, \quad \beta_1 = -\rho, \quad \rho = \frac{a}{A} r^A \quad \left(A = \frac{p-q}{2} + 1,\ a = \sqrt{\frac{g_1}{g_2}}\right)$$

$$\alpha_k = (-1)^k \frac{g_1 (2k-3+s)!!}{A(s-1)(k-1)!(k+s-1)!} r^{p+4} \rho^{k-1} \tag{3.8}$$

$$\beta_k = (-1)^k \frac{(s-2)(2k-3+s)!!}{(s-3)(k-2+s)!\,k!} \rho^k$$

$$(k = 0, 1, 2, \ldots; s = (p+4)/A \neq 1; 2; 3)$$

For an isotropic body $g_1 = g_2$, $p = q = 0$, $s = 4$, $\rho = r$. The coefficients $a_k$, $b_k$ can be calculated analogously from (3.6). Let us note that it follows from (3.8) that if there is a connection between $p$ and $q$ expressed by one of the equalities

$$(2n-1)A + p + 4 = 0 \qquad (n = 1, 2, \ldots) \tag{3.9}$$

then (2.2) become finite sums, and the question of convergence falls away. For example, if we set $n = 1$ in (3.9), we then arrive at the equalities

$$A = -(p+4), \qquad q = 3p + 10$$

In this case (1.4) and (2.2) have the form

$$\varphi = \operatorname{Im} \alpha_1 w', \quad \psi = \operatorname{Re}(\beta w + \beta_1 w'), \quad \rho = -\frac{a}{p+4} r^{-(p+4)} \tag{3.10}$$

It can be seen by direct substitution that the functions (3.10) satisfy the system (1.1).

**4. Torsion of a hollow inhomogeneous rod with mixed boundary conditions on the side surfaces.** Let us have a circular rod of length $l$ with coaxial cylindrical surfaces of radii $R_1$ and $R_2$ $(R_2 > R_1)$.

We consider the problem when stresses are given on one of the surfaces and displacements on the other in the form of functions of the coordinate $z$. The rod endfaces are stress-free, and the shear moduli are given in the form of piecewise-continuous functions of the radius. Taking account of (1.2), we have an internal mixed problem in a closed domain on one part of whose boundary Dirichlet conditions are specified for the displacement function, while we have the following Neumann conditions for the stress function on the other part:

$$\left.\frac{\partial \varphi}{\partial r}\right|_{z=0} = 0, \qquad \left.\frac{\partial \varphi}{\partial r}\right|_{z=l} = 0$$

$$v\big|_{r=R_1} = R_1 f_1(z) \tag{4.1}$$

$$\left.\frac{\partial \varphi}{\partial z}\right|_{r=R_2} = -R_2^2 f(z) = f_2(z)$$

Here $f_1(z)$, $f_2(z)$ are given piecewise-continuous functions of bounded variation in the interval $(0, l)$.

To solve the problem, let us use a linear integro-differential combination of (2.2) and (3.4)

$$\varphi = \varphi_1 + \varphi_2 = \operatorname{Im} \sum_{k=0}^{\infty} \left[ \alpha_k(r) w^{(k)}(\zeta) + a_k(r) \int f(\zeta)\, d\zeta^k \right] \tag{4.2}$$

$$v = r(\psi_1 + \psi_2) = r \operatorname{Re} \sum_{k=0}^{\infty} \left[ \beta_k(r) w^{(k)}(\zeta) + b_k(r) \int f(\zeta) \, d\zeta^k \right]$$

As the functions $w(\zeta)$ and $f(\zeta)$ we select the series

$$w(\zeta) = \sum_{n=1}^{\infty} A_n e^{-n\omega\zeta}, \qquad f(\zeta) = \sum_{n=1}^{\infty} B_n e^{-n\omega\zeta} \tag{4.3}$$

($A_n$, $B_n$ are arbitrary constants, $\omega$ is a fixed constant). We calculate the appropriate derivatives and integrals of (4.3) inserting them into (4.2). Hence, after extracting real and imaginary parts, we obtain

$$\varphi = \sum_{n=1}^{\infty} (\Delta_n^{\alpha}(r) A_n + \delta_n^{a}(r) B_n) \sin n\omega z$$

$$v = r \sum_{n=1}^{\infty} (\Delta_n^{\beta}(r) A_n + \delta_n^{b}(r) B_n) \cos n\omega z \tag{4.4}$$

Here the summation signs are permuted and the following notation is introduced.

$$\Delta_n^{\alpha}(r) = e^{-n\omega\rho} \sum_{k=0}^{\infty} (-1)^k (n\omega)^k \alpha_k(r)$$

$$\delta_n^{a}(r) = e^{-n\omega\rho} \sum_{k=0}^{\infty} (-1)^k \frac{a_k(r)}{(n\omega)^k} \tag{4.5}$$

($\Delta_n^{\beta}$ and $\delta_n^{b}$ are expressed by the same formulas (4.5) in which $\alpha_k$ and $a_k$ have just been replaced by $\beta_k$ and $b_k$, respectively). We set $\omega = \pi/l$ in (4.4). Then taking account of the first formula in (1.2), we note that the first two conditions in (4.1) are satisfied automatically. The other two conditions in (4.1) are also satisfied if the coefficients $A_n$, $B_n$ in the range $0 \leqslant z \leqslant l$ are determined from the conditions

$$\sum_{n=1}^{\infty} (\Delta_1^{\beta} A_n + \delta_1^{b} B_n) \cos n\omega z = f_1(z)$$

$$\sum_{n=1}^{\infty} (n\omega)(\Delta_2^{\alpha} A_n + \delta_2^{a} B_n) \cos n\omega z = f_2(z)$$

($\Delta_1^{\beta}$, $\delta_1^{b}$, $\Delta_2^{\alpha}$, $\delta_2^{a}$ are constants which we obtain if $R_1$ and $R_2$, respectively, are inserted into (4.5)). We expand the functions $f_1(z)$ and $f_2(z)$ in cosine Fourier series in the interval $(0, l)$, and then we use the ordinary Fourier method to find the coefficients $A_n$, $B_n$

$$A_n = \frac{1}{\Delta}(n\omega c_n \delta_2^{a} - d_n \delta_1^{b}), \qquad B_n = \frac{1}{\Delta}(d_n \Delta_1^{\beta} - n\omega c_n \Delta_2^{\alpha})$$

$$\Delta = n\omega(\Delta_1^{\beta}\delta_2^{a} - \Delta_2^{\alpha}\delta_1^{b}), \qquad n = 1, 2, \ldots \tag{4.6}$$

The coefficients $c_n$ and $d_n$ in (4.6) are found from the formulas

$$c_n = \frac{2}{l}\int_0^l f_1(z) \cos n\omega z \, dz, \qquad d_n = \frac{2}{l}\int_0^l f_2(z) \cos n\omega z \, dz \tag{4.7}$$

$$(n = 0, 1, 2, \ldots)$$

Hence $c_0 = d_0 = 0$, which results in specific conditions imposed on the functions $f_1(z)$ and $f_2(z)$.

As an illustration, let us consider the case when

$$f_1(z) = h_1 + h_2 z^2, \qquad f_2 = k_0 + k_1 z + k_2 z^2$$

($h_1$, $h_2$, $k_0$, $k_1$, $k_2$ are constants). Then we obtain from (4.7)

$$c_n = (-1)^n \frac{2h_2}{n\omega}, \qquad d_n = \frac{2}{n\omega l}\{[(-1)^n - 1] k_1 + (-1)^n l k_2\} \tag{4.8}$$

$$h_1 = -\frac{l^3 h_2}{3}, \qquad k_0 = -l\left(\frac{k_1}{2} + \frac{l^2 k_2}{3}\right) \tag{4.9}$$

The relationships (4.9) impose constraints on the coefficients $h$ and $k$.

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## ON SOME PROPERTIES OF EQUATIONS OF A MODEL OF COUPLED TERMOPLASTICITY

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Within the scope of models of elastic-plastic media, without taking account of thermal effects, the rates of change in the stresses are determined uniquely by means of a given state of stress and strain rates [1]. The constraint which should be imposed on a coupled thermoplasticity model so that the mentioned property would also exist in this case is considered herein. It is shown for the simplest coupled thermoplasticity model, that when heat conduction is neglected, there exists a domain of states of stress for which the system of plastic flow equations is not evolutionary, and also a domain of states of stress for which shock formation occurs from smooth initial conditions (reversing of simple waves). These properties can also be interpreted as the properties of an uncoupled plasticity model with a nongradient plastic flow law. An exam-